ISSN 0965-5425, Computational Mathematics and Mathematical Physics, 2017, Vol. 57, No. 3, pp. 453–463. © Pleiades Publishing, Ltd., 2017.
Original Russian Text © A.A. Kon'kov, 2017, published in Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki, 2017, Vol. 57, No. 3, pp. 448–458.

In memory of S.I. Pohozaev

Blow-up of Solutions for a Class of Nondivergence Elliptic Inequalities

A. A. Kon'kov
Faculty of Mechanics and Mathematics, Moscow State University, Moscow, 119991 Russia
e-mail: konkov@mech.math.msu.su
Received July 26, 2016

Abstract—Blow-up conditions are obtained for second-order nondivergence elliptic inequalities containing terms with lower order derivatives.

Keywords: elliptic inequalities, unbounded domains, blow-up.
DOI: 10.1134/S0965542517030083

1. INTRODUCTION

Let Ω be an unbounded open subset of $\mathbb{R}^n$, $n \geq 2$. We study nonnegative solutions of the problem

$$\sum_{i,j=1}^{n} a_{ij}(x,u)\frac{\partial^2 u}{\partial x_i \partial x_j} + b(x,u)|Du|^\alpha \geq f(x,u) \quad \text{in} \quad \Omega, \tag{1.1}$$

$$u|_{\partial\Omega} = 0, \tag{1.2}$$

where $D = (\partial_{x_1}, \dots, \partial_{x_n})$ is the gradient operator, b and f are nonnegative functions, $\alpha > 0$ is a real number, and the coefficients of the highest derivatives satisfy the condition

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j > 0$$

for all $x = (x_1, \dots, x_n) \in \Omega$, $t \in (0,\infty)$, and $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n \setminus \{0\}$. Assume also that

$$f(x,t) > p(|x|)g(t)\sum_{i,j=1}^{n} |a_{ij}(x,t)| + b(x,t)(|x|\,p(|x|))^\alpha h(t) \tag{1.3}$$

for all $x \in \Omega$ and $t \in (0,\infty)$, where $p : [0,\infty) \to [0,\infty)$ is a locally bounded measurable function and $g : (0,\infty) \to (0,\infty)$ and $h : (0,\infty) \to (0,\infty)$ are continuous functions.

A function $u \geq 0$ is said to be a nonnegative solution of problem (1.1), (1.2) if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and relations (1.1), (1.2) hold in the classical sense (see [1]). In the case of $\Omega = \mathbb{R}^n$, condition (1.2) is assumed to hold.

For every $\varphi : (0,\infty) \to \mathbb{R}$ and a real number $\sigma > 1$ we define

$$\varphi_\sigma(r) = \inf_{(r/\sigma,\sigma r)} \varphi.$$

We are interested in conditions guaranteeing the triviality of any nonnegative solution of problem (1.1), (1.2) which are also known as blow-up conditions. Similar issues for nonlinearities of the Emden–Fowler type were considered in [2–14]. The case of general nonlinearity for equations and inequalities that do not involve lower order derivatives was studied in [15–20]. For inequalities with general nonlinearity and lower order derivatives, sufficient blow-up conditions were obtained in [21, 22]. However, the dependence of coefficients of the differential operator on the function u was not taken into account in [21, 22]. Moreover, additional requirements were imposed in [22] on the growth of coefficients of lower order derivatives. Thus, results of these works cannot be applied to certain inequalities, specifically, to those presented in Examples 2.1–2.3 (see below).

2. MAIN RESULTS

Theorem 2.1. *Let*

$$\int_1^\infty (g_\theta(t)t)^{-1/2}dt < \infty,$$

$$\int_1^\infty h_\theta^{-1/\alpha}(t)dt < \infty,$$

$$\int_1^\infty rp_\sigma(r)dr < \infty,$$

where $\theta > 1$ *and* $\sigma > 1$ *are real numbers. Then any nonnegative solution of problem* (1.1), (1.2) *vanishes identically.*

Example 2.1. Consider the inequality

$$\Delta u + \beta(x)u^\mu |Du|^\alpha \ge \rho(x)u^\lambda \quad \text{in} \quad \mathbb{R}^n, \tag{2.1}$$

where $\beta : \Omega \to [0,\infty)$ and $\rho : \Omega \to (0,\infty)$ are locally bounded functions satisfying the relations

$$\beta(x) \le \beta_0 |x|^k \quad \text{and} \quad \rho(x) \ge \rho_0 |x|^l, \quad \beta_0, \rho_0 = \text{const} > 0, \tag{2.2}$$

for all x in a neighborhood of infinity. As before, we assume that $\alpha > 0$, while μ, λ, k, and l can be arbitrary real numbers.

Let $g(t) = t^\lambda$ and $h(t) = t^{\lambda-\mu}$. Then condition (1.3) holds for a locally bounded measurable function $p : [0,\infty) \to [0,\infty)$ such that

$$p(r) \sim r^{\min\{(l-k)/\alpha-1,l\}} \quad \text{as} \quad r \to \infty, \tag{2.3}$$

i.e.,

$$c_1 r^{\min\{(l-k)/\alpha-1,l\}} \le p(r) \le c_2 r^{\min\{(l-k)/\alpha-1,l\}}$$

for all r in a neighborhood of infinity, where $c_1 > 0$ and $c_2 > 0$ are constants. By Theorem 2.1, if

$$\lambda > \max\{1, \alpha + \mu\} \tag{2.4}$$

and

$$\min\{l - k + \alpha, l + 2\} \ge 0, \tag{2.5}$$

then any nonnegative solution of inequality (2.1) is identically zero.

Conditions (2.4) and (2.5) are sharp. Specifically, it can be shown that, if (2.4) is not satisfied, then inequality (2.1) has a positive solution for any locally bounded functions $\beta : \Omega \to [0,\infty)$ and $\rho : \Omega \to (0,\infty)$. In turn, if condition (2.5) is not satisfied, then there are locally bounded measurable functions $\beta : \Omega \to [0,\infty)$ and $\rho : \Omega \to (0,\infty)$ such that relations (2.2) hold and (2.1) has a positive solution.

Example 2.2. Let the locally bounded functions $\beta : \Omega \to [0,\infty)$ and $\rho : \Omega \to (0,\infty)$ in (2.1) satisfy the inequalities

$$\beta(x) \le \beta_0 |x|^k \ln^s |x| \quad \text{and} \quad \rho(x) \ge \rho_0 |x|^l \ln^m |x|, \quad \beta_0, \rho_0 = \text{const} > 0,$$

for all x from a neighborhood of infinity, where k, s, l, and m are real numbers and

$$\min\{l - k + \alpha, l + 2\} = 0.$$

In other words, we consider the case of critical exponents l and k in condition (2.5).

Following the preceding example, let $g(t) = t^{\lambda}$ and $h(t) = t^{\lambda-\mu}$. It is easy to see that (1.3) holds for a locally bounded measurable function $p : [0,\infty) \to [0,\infty)$ such that

$$p(r) \sim r^{-2} \ln^{\gamma} r \quad \text{as} \quad r \to \infty,$$

where

$$\gamma = \begin{cases} m, & l + 2 < l - k + \alpha, \\ \min\{(m - s)/\alpha, m\}, & l + 2 = l - k + \alpha, \\ (m - s)/\alpha, & l + 2 > l - k + \alpha. \end{cases}$$

Thus, by Theorem 2.1, if (2.4) holds and, additionally,

$$\gamma \geq -1, \tag{2.6}$$

then any nonnegative solution of inequality (2.1) vanishes identically.

As was said above, condition (2.4) is sharp. It can also be shown that (2.6) is sharp for $n \geq 3$.

Example 2.3. Consider the inequality

$$\Delta u + \beta(x) u^{\mu} |Du|^{\alpha} \geq \rho(x) u^{\lambda} \ln^{\nu}(1 + u) \quad \text{in} \quad \mathbb{R}^n, \tag{2.7}$$

where $\alpha > 0$ while $\beta : \Omega \to [0,\infty)$ and $\rho : \Omega \to (0,\infty)$ are locally bounded functions satisfying (2.2). Assume that

$$\lambda = \max\{1, \alpha + \mu\},$$

i.e., we are interested in the case of a critical exponent λ in condition (2.4).

Let $g(t) = t^{\lambda} \ln^{\nu}(1 + t)$ and $h(t) = t^{\lambda-\mu} \ln^{\nu}(1 + t)$. Then there is a locally bounded measurable function $p : [0,\infty) \to [0,\infty)$ such that (1.3) and (2.3) hold. Thus, by Theorem 2.1, for any nonnegative solution of inequality (2.7) to vanish identically, it is sufficient that (2.5) holds and, additionally,

$$\nu > \begin{cases} 2, & \alpha + \mu < 1, \\ \max\{2, \alpha\}, & \alpha + \mu = 1, \\ \alpha, & \alpha + \mu > 1. \end{cases} \tag{2.8}$$

These conditions are sharp. Indeed, if (2.8) is violated, then, as can be shown, (2.7) has a positive solution for all locally bounded functions $\beta : \Omega \to [0,\infty)$ and $\rho : \Omega \to (0,\infty)$. At the same time, if (2.5) does not hold, then (2.7) has a positive solution for some locally bounded measurable functions $\beta : \Omega \to [0,\infty)$ and $\rho : \Omega \to (0,\infty)$ satisfying relations (2.2).

3. PROOF OF THEOREM 2.1

We introduce the following notation. Let B_r^x and S_r^x denote the open ball and sphere in $\mathbb{R}^n$ of radius r centered at the point x. In the case of $x = 0$, we write B_r and S_r instead of B_r^0 and S_r^0, respectively.

By default, we assume that u is a nonnegative solution of problem (1.1), (1.2). For every $r \in (0,\infty)$ such that $S_r \cap \Omega \neq \emptyset$ let

$$M(r) = \sup_{S_r \cap \Omega} u.$$

In what follows, let θ and σ be the real numbers from the conditions of Theorem 2.1. By C we denote positive constants (possibly different) that can depend only on n, α, θ, and σ.

We need several preliminary results.

Lemma 3.1 (maximum principle). *Let ω be a nonempty open bounded subset of Ω; $v_1, v_2 \in C^2(\omega) \cap C(\overline{\omega})$; and, additionally,*

$$\sum_{i,j=1}^{n} a_{ij}(x,u) \frac{\partial^2 v_1}{\partial x_i \partial x_j} + b(x,u)|Dv_1|^{\alpha} > \sum_{i,j=1}^{n} a_{ij}(x,u) \frac{\partial^2 v_2}{\partial x_i \partial x_j} + b(x,u)|Dv_2|^{\alpha} \quad in \quad \omega,$$

$$v_1|_{\partial\omega} \leq v_2|_{\partial\omega}.$$

Then

$$v_1(x) \le v_2(x)$$

for all $x \in \omega$.

Proof. We follow the standard arguments (see [1]). The only delicacy is that the nonlinear term in the differential operator has to be taken into account. However, it can be seen that this obstacle is not essential. To avoid unfounded assertions, we present a complete proof.

Assume the opposite, namely, let

$$v_1(x) > v_2(x)$$

for some $x \in \omega$. Define

$$v = v_1 - v_2.$$

Since $v \in C(\overline{\omega})$ and $\overline{\omega}$ is a compact set, there is $\tilde{x} \in \overline{\omega}$ such that

$$v(\tilde{x}) = \sup_{\overline{\omega}} v > 0. \tag{3.1}$$

According to the conditions of the lemma, the function v is not positive on $\partial\omega$; therefore, $\tilde{x} \in \omega$, whence $Dv(\tilde{x}) = 0$ or, in other words, $Dv_1(\tilde{x}) = Dv_2(\tilde{x})$. Thus,

$$\sum_{i,j=1}^{n} a_{ij}(x,u) \frac{\partial^2 v}{\partial x_i \partial x_j}\bigg|_{x=\tilde{x}} > 0.$$

After changing to the variables $y = y(x)$ at the point $\tilde{x}$, the last inequality becomes

$$\sum_{i=1}^{n} \frac{\partial^2 v}{\partial y_i^2}\bigg|_{y=y(\tilde{x})} > 0,$$

so

$$\frac{\partial^2 v}{\partial y_i^2}\bigg|_{y=y(\tilde{x})} > 0$$

for some $1 \le i \le n$, which contradicts (3.1).

The lemma is completely proved.

Corollary 3.1. Let $S_r \cap \Omega \ne \emptyset$ for some $r \in (0,\infty)$. Then

$$M(r) = \sup_{B_r \cap \Omega} u. \tag{3.2}$$

If, additionally, $M(r) > 0$, then $S_{r'} \cap \Omega \ne \emptyset$ for all $r' \in [r,\infty)$ and M is a monotonically nondecreasing continuous function on $[r,\infty)$.

Proof. It can be assumed that

$$\sup_{B_r \cap \Omega} u > 0;$$

otherwise formula (3.2) is obvious. Let $\omega = \{x \in B_r \cap \Omega : u(x) > 0\}$. By virtue of condition (1.3), we have

$$\sum_{i,j=1}^{n} a_{ij}(x,u) \frac{\partial^2 u}{\partial x_i \partial x_j} + b(x,u)|Du|^{\alpha} > 0 \quad \text{in} \quad \omega.$$

Therefore, (3.2) follows from Lemma 3.1, where $v_1 = u$ and

$$v_2 = \sup_{S_r \cap \Omega} u$$

is a constant function.

Furthermore, if $M(r) > 0$ and $S_{r'} \cap \Omega = \emptyset$ for some $r' \in (r, \infty)$, then u vanishes on $\partial\omega'$, where $\omega' = \{x \in B_{r'} \cap \Omega : u(x) > 0\}$ is a nonempty open bounded set, so we obtain a contradiction to Lemma 3.1. The monotonicity of M follows straightforwardly from the fact that

$$M(r') = \sup_{B_{r'} \cap \Omega} u, \quad r' \in (r, \infty). \tag{3.3}$$

To prove the continuity of M, we extend u to the entire space $\mathbb{R}^n$ by setting $u(x) = 0$ for $x \in \mathbb{R}^n \setminus \Omega$. According to (1.2), the new function is continuous in $\mathbb{R}^n$; moreover, (3.3) can be rewritten as

$$M(r') = \sup_{B_{r'}} u, \quad r' \in (r, \infty).$$

Thus, we conclude that M is continuous on the interval $[r, \infty)$.

The proof is complete.

Lemma 3.2. *Let* $0 < r_1 < r_2$ *be real numbers and* $M(r_1) > 0$. *Then at least one of the following two inequalities holds:*

$$M(r_2) - M(r_1) \geq C(r_2 - r_1)^2 \inf_{(r_1, r_2)} p \inf_{(M(r_1), M(r_2))} g,$$

$$M(r_2) - M(r_1) \geq C(r_2 - r_1) r_1 \inf_{(r_1, r_2)} p \inf_{(M(r_1), M(r_2))} h^{1/\alpha}.$$

Proof. Consider $y \in S_{(r_1 + r_2)/2} \cap \Omega$ such that

$$u(y) = M\left(\frac{r_1 + r_2}{2}\right).$$

Furthermore, let $0 < \mu < M(r_1)$ be a real number and $\varphi \in C^\infty(\mathbb{R})$ be a monotonically nondecreasing function that vanishes identically on the interval $(-\infty, 1/3]$ and equals unity on $[1/2, \infty)$. Let $\omega = \{x \in B^y_{(r_2 - r_1)/2} \cap \Omega : u(x) > \mu\}$ and

$$w(x) = k\varphi\left(\frac{|x - y|}{r_2 - r_1}\right),$$

where

$$k = \frac{\min\{k_1, k_2\}}{\|\varphi\|_{C^2([1/3, 1/2])}},$$

$$k_1 = \frac{1}{4}(r_2 - r_1)^2 \inf_{(r_1, r_2)} p \inf_{(\mu, M(r_2))} g,$$

and

$$k_2 = (r_2 - r_1) r_1 \inf_{(r_1, r_2)} p \inf_{(\mu, M(r_2))} h^{1/\alpha}.$$

It is easy to see that

$$\sum_{i,j=1}^n a_{ij}(x, u) \frac{\partial^2 w}{\partial x_i \partial x_j} + b(x, u)|Dw|^\alpha = k\varphi''\left(\frac{|x - y|}{r_2 - r_1}\right) \sum_{i,j=1}^n \frac{a_{ij}(x, u)(x_i - y_i)(x_j - y_j)}{|x - y|^2 (r_2 - r_1)^2}$$

$$+ k\varphi'\left(\frac{|x - y|}{r_2 - r_1}\right) \sum_{i=1}^n \frac{a_{ii}(x, u)}{|x - y|(r_2 - r_1)} - k\varphi'\left(\frac{|x - y|}{r_2 - r_1}\right) \sum_{i,j=1}^n \frac{a_{ij}(x, u)(x_i - y_i)(x_j - y_j)}{|x - y|^3 (r_2 - r_1)}$$

$$+ b(x, u)\left|\frac{k}{r_2 - r_1} \varphi'\left(\frac{|x - y|}{r_2 - r_1}\right)\right|^\alpha,$$

whence we obtain

$$\sum_{i,j=1}^{n} a_{ij}(x,u)\frac{\partial^2 w}{\partial x_i \partial x_j} + b(x,u)|Dw|^{\alpha} \le p(|x|)g(u)\sum_{i,j=1}^{n}|a_{ij}(x,u)| + b(x,u)(|x|\,p(|x|))^{\alpha}h(u) \quad \text{in} \quad \omega. \tag{3.4}$$

Let us show that

$$\sup_{\partial\omega}(u-w) \ge \sup_{\omega}(u-w). \tag{3.5}$$

Indeed, if (3.5) does not hold, then there is a real number $\varepsilon > 0$ such that

$$\sup_{\partial\omega}(u-w) < \sup_{\omega}(u-w) - \varepsilon. \tag{3.6}$$

Define

$$v(x) = w(x) + \sup_{\omega}(u-w) - \varepsilon.$$

Taking into account (1.1), (1.3), and (3.4), we have

$$\sum_{i,j=1}^{n} a_{ij}(x,u)\frac{\partial^2 u}{\partial x_i \partial x_j} + b(x,u)|Du|^{\alpha} > \sum_{i,j=1}^{n} a_{ij}(x,u)\frac{\partial^2 v}{\partial x_i \partial x_j} + b(x,u)|Dv|^{\alpha} \quad \text{in} \quad \omega.$$

Moreover, it follows from (3.6) that

$$u|_{\partial\omega} < v|_{\partial\omega}.$$

Thus, by Lemma 3.1,

$$u(x) \le v(x)$$

for all $x \in \omega$ or, in other words,

$$\sup_{\omega}(u-w) \le \sup_{\omega}(u-w) - \varepsilon.$$

This contradiction proves (3.5).

Note that $y \in \omega$ and $w(y) = 0$. Thus,

$$\sup_{\omega}(u-w) \ge u(y) = M\left(\frac{r_1+r_2}{2}\right) \ge M(r_1) > \mu. \tag{3.7}$$

At the same time,

$$\sup_{B^y_{(r_2-r_1)/2}\cap\partial\omega}(u-w) \le \mu,$$

so (3.5) implies that

$$\sup_{S^y_{(r_2-r_1)/2}\cap\partial\omega}(u-w) \ge \sup_{\omega}(u-w),$$

whence, in view of

$$w|_{S^y_{(r_2-r_1)/2}} = k$$

and

$$\sup_{S^y_{(r_2-r_1)/2}\cap\partial\omega} u \le M(r_2),$$

we obtain

$$M(r_2) - k \ge \sup_{\omega}(u-w).$$

Combining this estimate with (3.7) yields

$$M(r_2) - M(r_1) \ge k.$$

Thus, at least one of the following two inequalities holds:

$$M(r_2) - M(r_1) \geq C(r_2 - r_1)^2 \inf_{(r_1,r_2)} p \inf_{(\mu, M(r_2))} g,$$

$$M(r_2) - M(r_1) \geq C(r_2 - r_1) r_1 \inf_{(r_1,r_2)} p \inf_{(\mu, M(r_2))} h^{1/\alpha}.$$

Passing to the limit as $\mu \to M(r_1) - 0$, we complete the proof.

Lemma 3.3. *Let $0 < r_1 < r_2$ be real numbers such that $r_2 \leq \sigma^{1/2} r_1$ and $M(r_2) \geq \theta^{1/2} M(r_1) > 0$. Then at least one of the following two inequalities is valid:*

$$\int_{M(r_1)}^{M(r_2)} (g_\theta(t)t)^{-1/2} dt \geq C \int_{r_1}^{r_2} p_{\sqrt{\sigma}}^{1/2}(r) dr, \tag{3.8}$$

$$\int_{M(r_1)}^{M(r_2)} h_\theta^{-1/\alpha}(t) dt \geq C \int_{r_1}^{r_2} r p_\sigma(r) dr. \tag{3.9}$$

Proof. Let k be the greatest positive integer satisfying the condition $M(r_2) \geq \theta^{k/2} M(r_1)$. Consider a sequence $\{\rho_i\}_{i=0}^k$, where $\rho_0 = r_1$, $\rho_k = r_2$, and $\rho_i \in (r_1, r_2)$, $i = 1, \ldots, k-1$, are chosen so that $M(\rho_i) = \theta^{1/2} M(\rho_{i-1})$. It can be seen that

$$\rho_i < \rho_{i+1} \quad \text{and} \quad \theta^{1/2} M(\rho_i) \leq M(\rho_{i+1}) < \theta M(\rho_i), \quad i = 0, \ldots, k-1.$$

By Lemma 3.2, for any integer $0 \leq i \leq k-1$ we have at least one of the inequalities

$$M(\rho_{i+1}) - M(\rho_i) \geq C(\rho_{i+1} - \rho_i)^2 \inf_{(\rho_i, \rho_{i+1})} p \inf_{(M(\rho_i), M(\rho_{i+1}))} g, \tag{3.10}$$

$$M(\rho_{i+1}) - M(\rho_i) \geq C(\rho_{i+1} - \rho_i)\rho_i \inf_{(\rho_i, \rho_{i+1})} p \inf_{(M(\rho_i), M(\rho_{i+1}))} h^{1/\alpha}. \tag{3.11}$$

Let Ξ_1 denote the set of integers $0 \leq i \leq k-1$ for which (3.10) holds, and let $\Xi_2 = \{0, 1, \ldots, k-1\} \setminus \Xi_1$. For any $i \in \Xi_1$, according to (3.10), we have

$$\left(\frac{M(\rho_{i+1}) - M(\rho_i)}{\inf\limits_{(M(\rho_i), M(\rho_{i+1}))} g} \right)^{1/2} \geq C(\rho_{i+1} - \rho_i) \inf_{(\rho_i, \rho_{i+1})} p^{1/2}.$$

Combining this inequality with

$$\int_{M(\rho_i)}^{M(\rho_{i+1})} (g_\theta(t)t)^{-1/2} dt \geq C \left(\frac{M(\rho_{i+1}) - M(\rho_i)}{\inf\limits_{(M(\rho_i), M(\rho_{i+1}))} g} \right)^{1/2}$$

yields the estimate

$$\int_{M(\rho_i)}^{M(\rho_{i+1})} (g_\theta(t)t)^{-1/2} dt \geq C(\rho_{i+1} - \rho_i) \inf_{(\rho_i, \rho_{i+1})} p^{1/2}. \tag{3.12}$$

Similarly, for any $i \in \Xi_2$, taking into account (3.11), we obtain

$$\frac{M(\rho_{i+1}) - M(\rho_i)}{\inf\limits_{(M(\rho_i), M(\rho_{i+1}))} h^{1/\alpha}} \geq C(\rho_{i+1} - \rho_i)\rho_i \inf_{(\rho_i, \rho_{i+1})} p,$$

whence, in view of

$$\int_{M(\rho_i)}^{M(\rho_{i+1})} h_\theta^{-1/\alpha}(t)dt \ge \frac{M(\rho_{i+1}) - M(\rho_i)}{\inf\limits_{(M(\rho_i), M(\rho_{i+1}))} h^{1/\alpha}}, \tag{3.13}$$

we have

$$\int_{M(\rho_i)}^{M(\rho_{i+1})} h_\theta^{-1/\alpha}(t)dt \ge C(\rho_{i+1} - \rho_i)\rho_i \inf_{(\rho_i, \rho_{i+1})} p. \tag{3.14}$$

If

$$\sum_{i \in \Xi_1} (\rho_{i+1} - \rho_i) \ge \frac{r_2 - r_1}{2}, \tag{3.15}$$

then, summing (3.12) over all $i \in \Xi_1$, we conclude that

$$\int_{M(r_1)}^{M(r_2)} (g_\theta(t)t)^{-1/2} dt \ge C(r_2 - r_1) \inf_{(r_1, r_2)} p^{1/2},$$

which straightforwardly yields (3.8).

On the other hand, if (3.15) does not hold, then

$$\sum_{i \in \Xi_2} (\rho_{i+1} - \rho_i) \ge \frac{r_2 - r_1}{2}.$$

Thus, summing (3.14) over all $i \in \Xi_2$, we obtain

$$\int_{M(r_1)}^{M(r_2)} h_\theta^{-1/\alpha}(t)dt \ge C(r_2 - r_1) r_1 \inf_{(r_1, r_2)} p,$$

whence, in turn, (3.9) follows.

The proof is completed.

Lemma 3.4. *Let* $0 < r_1 < r_2$ *be real numbers such that* $r_2 \ge \sigma^{1/4} r_1$ *and* $0 < M(r_2) \le \theta^{1/2} M(r_1)$. *Then*

$$\int_{M(r_1)}^{M(r_2)} \frac{dt}{g_{\sqrt{\theta}}(t)} \ge C \int_{r_1}^{r_2} r p_\sigma(r) dr \tag{3.16}$$

or (3.9) *is valid.*

Proof. Let k be a positive integer such that $r_2 \ge \sigma^{k/4} r_1$ and $r_2 < \sigma^{(k+1)/4} r_1$. Let $\rho_i = \sigma^{i/4} r_1$ for $i = 0, \ldots, k-1$, and $\rho_k = r_2$. It is easy to see that

$$\sigma^{1/4} \rho_i \le \rho_{i+1} < \sigma^{1/2} \rho_i, \quad i = 0, \ldots, k-1. \tag{3.17}$$

By Lemma 3.2, for any integer $0 \le i \le k-1$ at least one of inequalities (3.10), (3.11) is valid. As in the preceding lemma, let Ξ_1 denote the set of integers $0 \le i \le k-1$ for which (3.10) holds, and let $\Xi_2 = \{0, 1, \ldots, k-1\} \setminus \Xi_1$.

For any $i \in \Xi_1$ it follows from (3.10) and (3.17) that

$$\frac{M(\rho_{i+1}) - M(\rho_i)}{\inf\limits_{(M(\rho_i), M(\rho_{i+1}))} g} \ge C(\rho_{i+1} - \rho_i)\rho_i \inf_{(\rho_i, \rho_{i+1})} p.$$

Combining this inequality with the obvious estimates

$$\int_{M(\rho_i)}^{M(\rho_{i+1})} \frac{dt}{g_{\sqrt{\theta}}(t)} \geq \frac{M(\rho_{i+1}) - M(\rho_i)}{\inf\limits_{(M(\rho_i), M(\rho_{i+1}))} g}$$

and

$$(\rho_{i+1} - \rho_i)\rho_i \inf_{(\rho_i, \rho_{i+1})} p \geq C \int_{\rho_i}^{\rho_{i+1}} r p_\sigma(r) dr, \tag{3.18}$$

we derive

$$\int_{M(\rho_i)}^{M(\rho_{i+1})} \frac{dt}{g_{\sqrt{\theta}}(t)} \geq C \int_{\rho_i}^{\rho_{i+1}} r p_\sigma(r) dr. \tag{3.19}$$

Similarly, if $i \in \Xi_2$, then, according to (3.11), we have

$$\frac{M(\rho_{i+1}) - M(\rho_i)}{\inf\limits_{(M(\rho_i), M(\rho_{i+1}))} h^{1/\alpha}} \geq C(\rho_{i+1} - \rho_i)\rho_i \inf_{(\rho_i, \rho_{i+1})} p.$$

By (3.13) and (3.18), this yields

$$\int_{M(\rho_i)}^{M(\rho_{i+1})} h_\theta^{-1/\alpha}(t) dt \geq C \int_{\rho_i}^{\rho_{i+1}} r p_\sigma(r) dr. \tag{3.20}$$

Assume that

$$\sum_{i \in \Xi_1} \int_{\rho_i}^{\rho_{i+1}} r p_\sigma(r) dr \geq \frac{1}{2} \int_{r_1}^{r_2} r p_\sigma(r) dr. \tag{3.21}$$

Then, summing (3.19) over all $i \in \Xi_1$, we obtain inequality (3.16). If (3.21) does not hold, then

$$\sum_{i \in \Xi_2} \int_{\rho_i}^{\rho_{i+1}} r p_\sigma(r) dr \geq \frac{1}{2} \int_{r_1}^{r_2} r p_\sigma(r) dr.$$

Thus, summing (3.20) over all $i \in \Xi_2$, we arrive at (3.9).

The proof is completed.

Lemma 3.5. *Let* $0 < \varkappa \leq 1$, $\mu > 1$, $\nu > 1$, $M_1 > 0$, *and* $M_2 > 0$ *be real numbers and* $M_2 \geq \nu M_1$. *Then*

$$\left(\int_{M_1}^{M_2} \psi_\mu^{-\varkappa}(s) s^{\varkappa - 1} ds \right)^{1/\varkappa} \geq A \int_{M_1}^{M_2} \frac{ds}{\psi(s)}$$

for any measurable function $\psi : (0, \infty) \to (0, \infty)$ *such that* $\psi_\mu(s) > 0$ *for all* $s \in (0, \infty)$, *where* $A > 0$ *is a constant depending only on* μ, ν, *and* $\varkappa$.

Lemma 3.6. *Let* $0 < \varkappa \leq 1$, $\mu > 1$, $\nu > 1$, $r_1 > 0$, *and* $r_2 > 0$ *be real numbers and* $r_2 \geq \nu r_1$. *Then*

$$\left(\int_{r_1}^{r_2} \varphi^\varkappa(r) dr \right)^{1/\varkappa} \geq A \int_{r_1}^{r_2} r^{1/\varkappa - 1} \varphi_\mu(r) dr$$

for any measurable function $\varphi : [r_1, r_2] \to [0, \infty)$, *where* $A > 0$ *is a constant depending only on* μ, ν, *and* $\varkappa$.

Lemmas 3.5 and 3.6 were proved in [16, Lemmas 2.3 and 2.6].

Proof of Theorem 2.1. Assume the opposite, i.e., $M(r_0) > 0$ for some real number $r_0 > 0$, and let $r_i = \sigma^{i/2} r_0$, $i = 1, 2, \ldots$. By Lemmas 3.3 and 3.4, for any nonnegative integer i at least one of the following three inequalities is valid:

$$\int_{M(r_i)}^{M(r_{i+1})} (g_\theta(t)t)^{-1/2} dt \ge C \int_{r_i}^{r_{i+1}} p_{\sqrt{\sigma}}^{1/2}(r)dr, \tag{3.22}$$

$$\int_{M(r_i)}^{M(r_{i+1})} \frac{dt}{g_{\sqrt{\theta}}(t)} \ge C \int_{r_i}^{r_{i+1}} r p_\sigma(r) dr, \tag{3.23}$$

$$\int_{M(r_i)}^{M(r_{i+1})} h_\theta^{-1/\alpha}(t) dt \ge C \int_{r_i}^{r_{i+1}} r p_\sigma(r) dr. \tag{3.24}$$

Let Ξ_1, Ξ_2, and Ξ_3 denote the sets of nonnegative integers satisfying (3.22)–(3.24), respectively.

Summing (3.22) over all $i \in \Xi_1$ yields

$$\int_{M(r_0)}^{\infty} (g_\theta(t)t)^{-1/2} dt \ge C \sum_{i \in \Xi_1} \int_{r_i}^{r_{i+1}} p_{\sqrt{\sigma}}^{1/2}(r)dr.$$

Combining this with the estimate

$$\left(\sum_{i \in \Xi_1} \int_{r_i}^{r_{i+1}} p_{\sqrt{\sigma}}^{1/2}(r)dr \right)^2 \ge \sum_{i \in \Xi_1} \left(\int_{r_i}^{r_{i+1}} p_{\sqrt{\sigma}}^{1/2}(r)dr \right)^2 \ge C \sum_{i \in \Xi_1} \int_{r_i}^{r_{i+1}} r p_\sigma(r) dr,$$

which follows from Lemma 3.6, we conclude that

$$\left(\int_{M(r_0)}^{\infty} (g_\theta(t)t)^{-1/2} dt \right)^2 \ge C \sum_{i \in \Xi_1} \int_{r_i}^{r_{i+1}} r p_\sigma(r) dr. \tag{3.25}$$

Summing (3.23) over all $i \in \Xi_2$ yields

$$\int_{M(r_0)}^{\infty} \frac{dt}{g_{\sqrt{\theta}}(t)} \ge C \sum_{i \in \Xi_2} \int_{r_i}^{r_{i+1}} r p_\sigma(r) dr.$$

At the same time, by Lemma 3.5,

$$\left(\int_{M(r_0)}^{\infty} (g_\theta(t)t)^{-1/2} dt \right)^2 \ge C \int_{M(r_0)}^{\infty} \frac{dt}{g_{\sqrt{\theta}}(t)};$$

therefore,

$$\left(\int_{M(r_0)}^{\infty} (g_\theta(t)t)^{-1/2} dt \right)^2 \ge C \sum_{i \in \Xi_2} \int_{r_i}^{r_{i+1}} r p_\sigma(r) dr. \tag{3.26}$$

In an entirely similar manner, (3.24) implies the inequality

$$\int_{M(r_0)}^{\infty} h_\theta^{-1/\alpha}(t) dt \ge C \sum_{i \in \Xi_3} \int_{r_i}^{r_{i+1}} r p_\sigma(r) dr.$$

Combining it with (3.25) and (3.26), we have

$$\left(\int_{M(r_0)}^{\infty} (g_\theta(t)t)^{-1/2}dt\right)^2 + \int_{M(r_0)}^{\infty} h_\theta^{-1/\alpha}(t)dt \geq C\int_{r_0}^{\infty} rp_\sigma(r)dr,$$

which contradicts the conditions of the theorem.

The proof is completed.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research, project no. 11-01-12018-ofi-m-2011.

REFERENCES

1. E. M. Landis, *Second-Order Equations of Elliptic and Parabolic Type* (Nauka, Moscow, 1971; Am. Math. Soc., Providence, R.I., 1998).
2. I. Astashova, "On power and nonpower asymptotic behavior of positive solutions to Emden–Fowler type higher-order equations," Adv. Difference Equations Springer Open J., No. 2013:220 (2013).
3. I. Astashova, "On quasi-periodic solutions to a higher-order Emden–Fowler type differential equation," Boundary Value Probl., No. 2014:174 (2014).
4. I. V. Astashova, E. S. Karulina, S. S. Ezhak, M. Yu. Tel'nova, V. A. Nikishkin, and A. V. Filinovskii, *Qualitative Properties of Solutions to Differential Equations and Related Issues of Spectral Analysis* (YuNITI-DANA, Moscow, 2012) [in Russian].
5. L. Veron, "Comportement asymptotique des solutions d'equations elliptiques semi-lineaires dans R^n," Ann. Math. Pure. Appl. **127**, 25–50 (1981).
6. E. I. Galakhov, "Positive solutions of quasilinear elliptic equations," Math. Notes **78** (2), 185–193 (2005).
7. E. I. Galakhov, "On some partial differential inequalities with gradient terms," Proc. Steklov Inst. Math. **283**, 35–43 (2013).
8. V. A. Kondrat'ev and E. M. Landis, "On qualitative properties of solutions of a nonlinear equation of second order," Math. USSR-Sb. **63** (2), 337–350 (1989).
9. V. A. Kondrat'ev and S. L. Eidel'man, "On positive solutions to second-order quasilinear elliptic equations," Dokl. Akad. Nauk SSSR **334** (4), 427–428 (1994).
10. M. O. Korpusov, "Blow-up of the solution to a nonlocal equation with gradient nonlinearity," Vestn. Yuzhno-Ural. Gos. Univ., Ser. Mat. Model. Program., No. 11, 43–53 (2012).
11. M. O. Korpusov, "Solution blowup for the heat equation with double nonlinearity," Theor. Math. Phys. **172** (3), 1173–1176 (2012).
12. E. L. Mitidieri and S. I. Pohozaev, "Nonexistence of positive solutions for quasilinear elliptic problems on R^n," Proc. Steklov Inst. Math. **227**, 186–216 (1999).
13. E. L. Mitidieri and S. I. Pohozaev, "A priori estimates and blow-up of solutions to partial differential equations and inequalities," Proc. Steklov Inst. Math. **234**, 1–362 (2001).
14. Y. Naito and H. Usami, "Nonexistence results of positive entire solutions for quasilinear elliptic inequalities," Can. Math. Bull. **40**, 244–253 (1997).
15. J. B. Keller, "On solution of $\Delta u = f(u)$," Commun. Pure. Appl. Math. **10** (4), 503–510 (1957).
16. A. A. Kon'kov, "On solutions of nonautonomous ordinary differential equations," Izv. Math. **65** (2), 285–327 (2001).
17. Y. Naito and H. Usami, "Entire solutions of the inequality $\operatorname{div}(A(|Du|)Du) = f(u)$," Math. Z. **225**, 167–175 (1997).
18. R. Osserman, "On the inequality $\Delta u \geq f(u)$," Pacific J. Math. **7** (4), 1641–1647 (1957).
19. R. Filippucci, P. Pucci, and M. Rigoli, "Nonexistence of entire solutions of degenerate elliptic inequalities with weights," Arch. Ration. Mech. Anal. **188**, 155–179 (2008); Erratum, **188**, 181 (2008).
20. M. Ghergu and V. Radulescu, "Existence and nonexistence of entire solutions to the logistic differential equation," Abstr. Appl. Anal. **17**, 995–1003 (2003).
21. A. A. Kon'kov, "On properties of solutions of quasilinear second-order elliptic inequalities," Nonlinear Anal. **123–124**, 89–114 (2015).
22. R. Filippucci, P. Pucci, and M. Rigoli, "On entire solutions of degenerate elliptic differential inequalities with nonlinear gradient terms," J. Math. Anal. Appl. **356**, 689–697 (2009).

Translated by I. Ruzanova